$$Z(a)^n = \frac{Z}{Z-a} , \quad Z\{(a)^n - \delta_n\} = Z a^n - Z \delta_n$$

$$= \frac{Z}{Z-a} - 1 = \frac{a}{Z-a}$$

$$\therefore \boxed{Z(a^n - \delta_n) = \frac{a}{Z-a}}$$

## Solve the difference equation by the Z-transform :-

The Solution of difference equation by the Z-transform method is very useful, as in the solution of differential equation by Laplace transform essentially by using Z-transform. We transform difference equation into the algebric equation in Z. In the following we shall use the simplifier notation $x(n)$ to denote $X[nT]$.

$$Z[x(n+1)] = Z X(z) - Z x(0) \quad \text{if } x(0) = 0 \text{ then } Z x(n+1)$$

$$= Z x(z) \quad \text{where } x(z) = Z x(n)$$

$$Z[x(n+2)] = Z^2 x(z) - z^2 x(0) - z\, x(1)$$

$$Z[x(n+3)] = Z^3 x(z) - z^3 x(0) - z^2 x(1) - z\, x(2)$$

EX find the response $x(n)$ of the following system

$$x(n+2)-3x(n+1)+2x(n)=\delta(n) \quad \text{at } x(0)=x(1)=0$$

$$Z^2X(Z)-Z^2x(0)-Zx(1)-3ZX(Z)+3Zx(0)+2X(Z)=1$$

$$X(Z)[Z^2-3Z+2]=1$$

$$X(Z)=\frac{1}{Z^2-3Z+2}$$

$$Res_1[X(Z)\cdot Z^{n-1}]=\left[\frac{1}{(Z-1)(Z-2)}(Z-1)Z^{n-1}\right]\Bigg|_{Z=1}=\frac{Z^n}{Z(Z-2)}\Bigg|_{Z=1}=-(1)^n$$

$$Res_2[X(Z)\cdot Z^{n-1}]=\frac{1}{(Z-1)(Z-2)}(Z-2)Z^{n-1}\Bigg|_{Z=2}=(2)^{n-1}$$

$$\therefore\ X(Z)=-(1)^n+(2)^{n-1}$$

## Application in the discrete-time Fourier transform (DTFT)

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jnw}$$

However, in order for this series to converge, it is necessary that the signal be absolutely summable. Unfortunately, many of the signals that we would like to consider are not absolutely summable and, therefore, do not have a DTFT. Some examples include

$$x(n) = u(n) \qquad x(n) = (0.5)^n u(-n) \qquad x(n) = \sin n w_0$$

The Z-transform is a generalization of the DTFT that allows one to deal with such sequences and is defined as follows:

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

where $Z = re^{jw}$ is a complex variable. The values of $Z$ for which the sum of converges define a region in the Z-plane referred to as the "region of convergence (ROC)".

$$x(n) \overset{Z}{\longleftrightarrow} X(Z)$$

The Z-transform may be viewed as the DTFT of an exponentially weighted sequence. ~~Specially~~ Specifically, note that with $Z = re^{jw}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} [r^{-n} x(n)]\, e^{-jnw}$$

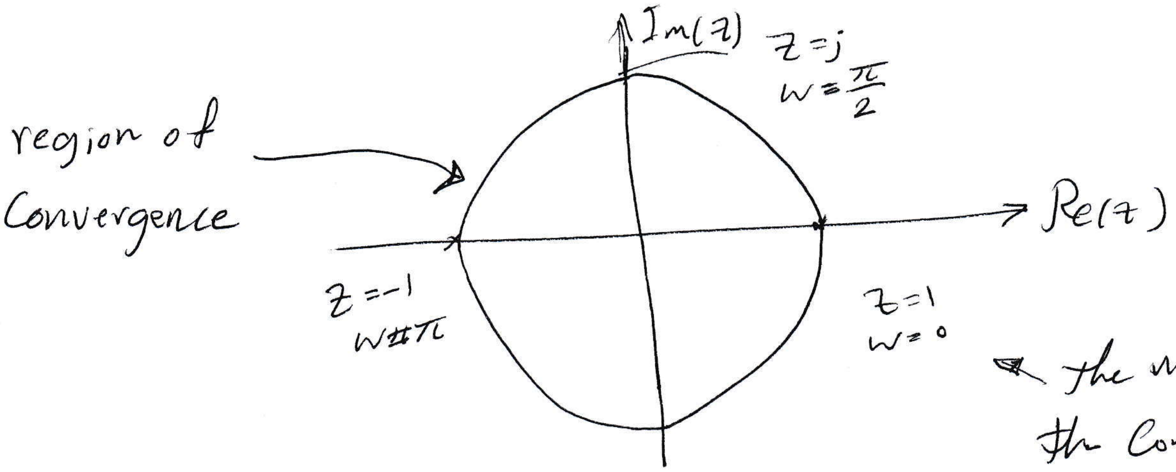
$X(z)$ is the discrete-time Fourier transform of the sequence $r^{-n} x(n)$. Furthermore, the ROC is determined by the range of values of "$r$" for which

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

$$Z = Re(Z) + j\, Im(Z) = r e^{jw}$$

the axes of the Z-plane are the real and imaginary parts of "Z" as in figure below, and the contour corresponding to $|Z| = 1$ is a circle of unit radius referred to as the "Unit Circle". the Z-transform evaluated on the unit circle corresponds to the DTFT.

$$X(e^{jw}) = X(z)\big|_{z = e^{jw}}$$




the unit circle in the Complex Z-plane

$X(e^{jw})$ for $0 \leq w \leq \pi$.

$$X(Z) = \frac{B(Z)}{A(Z)} = \frac{\sum_{K=0}^{q} b(K) Z^{-K}}{\sum_{K=0}^{P} a(K) Z^{-K}}$$

Factoring the numerator and denomenator polynomials, a rotational Z-transform may be expressed as follows:-

$$X(Z) = C \frac{\prod_{K=1}^{q} (1-\beta_K Z^{-1})}{\prod_{K=1}^{P} (1-\alpha_K Z^{-1})}$$

the roots of the numerator polynomial, $\beta_K$, are referred to as the "zero" of $X(Z)$, and the roots of the denomenator polynomial, $\alpha_K$ are referred to as the "pole", with the region of a convergence indicated by shading the appropriate region of the Z-plane the region of convergence is, in general can "annulus" of the form

$$\alpha < |Z| < \beta$$

1- A right-sided sequence has a Z-transform with a region of convergence that is the "exterior" of a circle:

$$ROC: |Z| > \alpha$$

2- A left-sided sequence has a Z-transform with a region of convergence that is the "interior" of a circle:

$$ROC: |Z| < \beta$$

EX Find the Z-transform of the sequence $x(n) = \alpha^n u(n)$. Using the definition of the Z-transform and the geometric series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

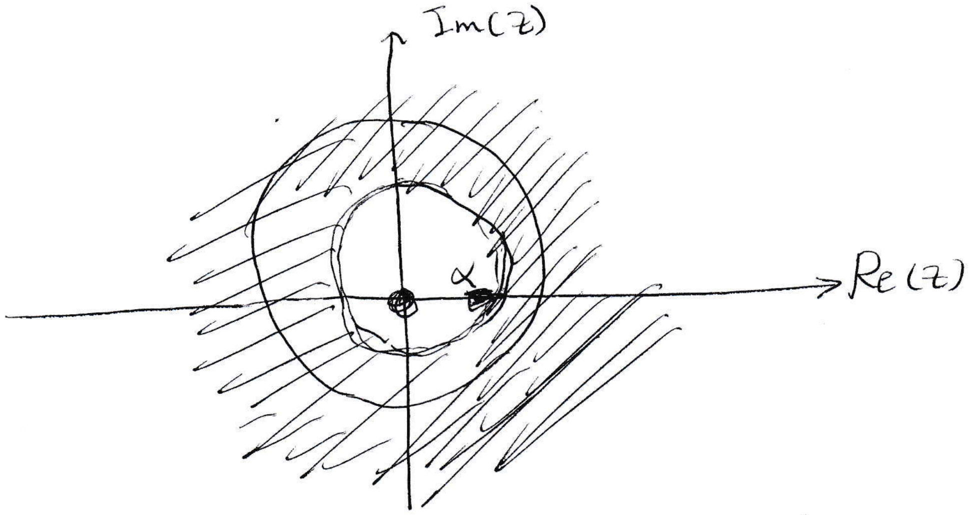
$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}$$

with the Convergence $|\alpha z^{-1}| < 1$. the ROC is the "exterior" of a circle defined by

$$|z| > |\alpha|$$

$$\boxed{X(z) = \frac{z}{z-\alpha}}$$

$X(s)$ has a zero at $z=0$ and a pole at $z=\alpha$



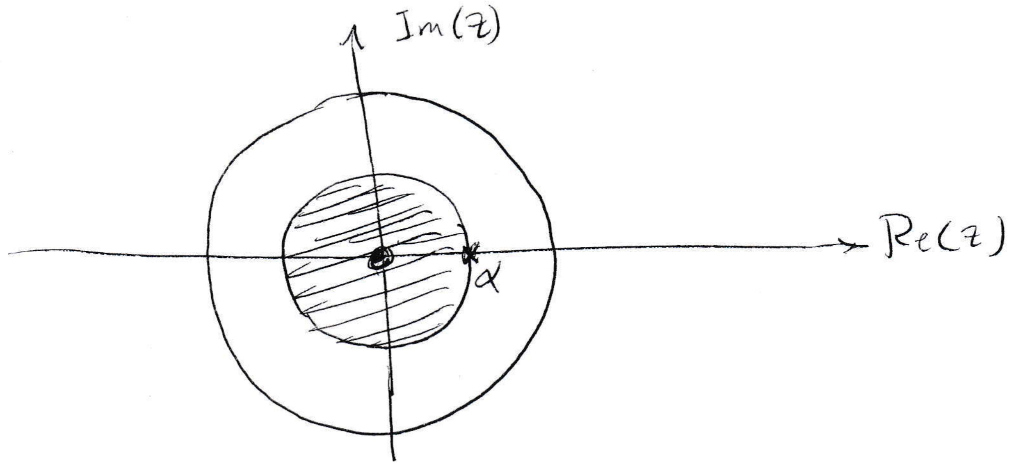

Unit Circle with the region of Convergence and the DTFT of $x(n)$ exist

Ex find the Z-transform of the sequence $x(n) = -a^{n} u(-n-1)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = -\sum_{n=-\infty}^{-1} a^{n} z^{-n} = -\sum_{n=0}^{\infty} (a^{-1} z)^{n+1}$$

$$= -\alpha^{-1} z \sum_{n=0}^{\infty} (a^{-1} z)^{n} = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = \frac{1}{1-\alpha z^{-1}}$$

$|1-\alpha z^{-1}| < 1$ or $|z| < |\alpha|$




Ex find the Z-transform of $x(n) = (\frac{1}{2})^{n} u(n) - 2^{n} u(-n-1)$.

for $x_1(n) = (\frac{1}{2})^{n} u(n)$ is

$$X_1(z) = \frac{1}{1-\frac{1}{2} z^{-1}} \qquad |z| > \frac{1}{2}$$

for $x_2(n) = -2^{n} u(-n-1)$ is

$$X_2(z) = \frac{1}{1-2 z^{-1}} \qquad |z| < 2$$

$x(n) = x_1(n) + x_2(n)$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} = \frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{z^{-1}})(1-2z^{-1})}$$

## Common Z-transform Pairs

| Sequence | Z-transform | Region of Convergence |
|---|---|---|
| $\delta(n)$ | $1$ | all $z$ |
| $\alpha^n u(n)$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| $-\alpha^n u(-n-1)$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| $n\alpha^n u(n)$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| $-n\alpha^n u(-n-1)$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| $\cos(nw_0)\, u(n)$ | $\frac{1-(\cos w_0)z^{-1}}{1-2(\cos w_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| $\sin(nw_0)\, u(n)$ | $\frac{(\sin w_0)\, z^{-1}}{1-2(\cos w_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |

## Time Reversal

If $x(n)$ has a Z-transform $X(z)$ with a region of Convergence $R_x$ that is the annulus $\alpha < |z| < \beta$, the Z-transform of the time-reversed sequence $x(-n)$ is

$$x(-n) \overset{Z}{\longleftrightarrow} X(z^{-1})$$

and has a region of Convergence $1/\beta < |z| < 1/\alpha$, which is denoted by $1/R_x$.

## Convolution theorem

it the most important Z-transform property ~~which~~ states that Convolution in the time domain is mapped into multiplication in the frequency domain, that is,

$$y(n) = x(n) * h(n) \overset{Z}{\longleftrightarrow} Y(z) = X(z)\,H(z).$$

EX we have

$$x(n) = \alpha^n u(n) \quad , \quad h(n) = \delta(n) - \alpha\,\delta(n-1)$$

the Z-transform of $x(n)$ is

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha|$$

the Z-transform of $h(n)$ is

$$H(z) = 1 - \alpha z^{-1} \qquad 0 < |z|$$

the Z-transform of the convolution $x(n)$ with $h(n)$ is

$$Y(z) = X(z)\,H(z) = \frac{1}{1-\alpha z^{-1}}\,(1-\alpha z^{-1}) = 1$$

EX find the Z-transform of $x(n) = n\alpha^n u(-n)$.

To find $X(z)$, use the time-reversal and derivative properties.

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} \qquad |z| > \alpha$$

$$\alpha^n u(-n) \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha^{-1} z} \qquad |z| < |\alpha|$$

using the derivative because we have $n\alpha^n u(-n)$

$$-z\frac{d}{dz}\,\frac{1}{1-\alpha^{-1}z} = -\frac{\alpha^{-1}z}{(1-\alpha^{-1}z)^2} \qquad |z| < \alpha$$

## Z-transform                Homework 1

1- $Z\, e^{-at} \sin wt$

2- $Z\, e^{-at} \cos wt$

3- $Z \cos wt$

4- $Z\, t \sin wt$

5- $Z\, t \cos wt$

6- $Z\, t\, e^{-at} \sin wt$

7- $Z\, t^2 e^{-at} \sin wt$

8- $Z\, e^{-(t+3T)}$

9- $Z\, e^{-(t-3T)}$

Z-transform    Homework 2

Q1 Find $Z^{-1}\dfrac{3}{2Z-6Z+6Z^{-1}}$

Q2 Find the $Z^{-1}$ or $x(n)$ for

$$X(Z)=\frac{4-\frac{7}{4}Z^{-1}+\frac{1}{4}Z^{-2}}{1-\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}$$

Q3 Consider the linear constant coefficient difference equation

$$y(n)=0.25\,y(n-2)+x(n),$$

$$x(n)=\delta(n-1) \text{ with } y(-1)=y(-2)=1$$

$\underline{Q_3}$ Find the Z-transform of each of the following sequences:

a) $x(n) = 2^n u(n) + 3\left(\frac{1}{2}\right)^n u(n)$

b) $x(n) = \cos(n\omega_0)\, u(n)$

a)

$$\cancel{\times} \quad 2^n u(n) \longleftrightarrow \frac{1}{1-2z^{-1}}, \quad |z| > |2|$$

$$3\left(\tfrac{1}{2}\right)^n u(n) \longleftrightarrow \frac{3}{1-\frac{1}{2}z^{-1}} \quad |z| > |\tfrac{1}{2}|$$

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1-\frac{1}{2}z^{-1}+3-6z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{4-\frac{13}{2}z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

(b) $x(n) = \cos(n\omega_0)\, u(n)$

$$= \frac{1}{2}\left[e^{jn\omega_0} + e^{-jn\omega_0}\right] u(n)$$

$$X(z) = \frac{1}{2}\,\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\,\frac{1}{1-e^{-j\omega_0}z^{-1}}$$

$$X(z) = \frac{1}{2}\,\frac{1-e^{-j\omega_0}z^{-1}+1-e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}$$

$$X(z) = \frac{1}{2} \frac{2 - z^{-1}(e^{jw_0} + e^{-jw_0})}{1 - e^{-jw_0} z^{-1} - e^{jw_0} z^{-1} + e^{jw_0 - jw_0} z^{-2}}$$

$$= \frac{1}{2} \frac{2 - 2z^{-1}\cos w_0}{1 - 2z^{-1}\cos w_0 + z^{-2}}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}(1 - \cos w_0 z^{-1})}{1 - 2\cos w_0 z^{-1} + z^{-2}}$$

$$X(z) = \frac{1 - (\cos w_0) z^{-1}}{1 - 2(\cos w_0) z^{-1} + z^{-2}} \qquad |z| > 1$$

**Q4** Find the Z-transform of each of the following sequences.

a) $x(n) = \left(\frac{1}{3}\right)^n u(-n)$

b) $x(n) = \left(\frac{1}{2}\right)^n u(n+2) + (3)^n u(-n-1)$

c) $x(n) = \left(\frac{1}{3}\right)^n \cos(nw_0)\, u(n)$

d) $x(n) = \alpha^{|n|}$

a)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} 3^n z^n = \frac{1}{1-3z}$$

$|3z| < 1$ or $|z| < \frac{1}{3}$

the time-reversed sequence $y(n) = x(-n) = (\frac{1}{3})^{-n} u(n)$
has a Z-transform given by

$$Y(z) = \frac{1}{1-3z^{-1}}$$

ROC $|z| > 3$

b)

$$(\frac{1}{2})^n u(n+2) = (\frac{1}{2})^n * 4 * (\frac{1}{2})^2 \; u(n+2)$$
$$= 4 \; (\frac{1}{2})^{n+2} u(n+2)$$

the Z-transform of $(\frac{1}{2})^n u(n+2)$ is $4z^2$ times the Z-transform of $(\frac{1}{2})^n u(n)$, that is,

$$(\frac{1}{2})^n u(n+2) \overset{z}{\longleftrightarrow} \frac{4z^2}{1-\frac{1}{2}z^{-1}}$$

ROC $|z| > \frac{1}{2}$

$$3^n u(-n-1) \overset{z}{\longleftrightarrow} -\frac{1}{1-3z^{-1}}$$

ROC $|z| \cancel{>} 3.$

the Z-transform is

$$X(z) = \frac{4z^2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-3z^{-1}}$$

ROC $\frac{1}{2} < |z| < 3$.

c)

the Z-transform of $\cos(n\omega_0)u(n)$ is

$$\cos(n\omega_0)u(n) \xleftrightarrow{Z} \frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}} \; ; \; |z|>1$$

by using the exponential property,

$$\alpha^n x(n) \xleftrightarrow{Z} X(\alpha^{-1}z)$$

$$\left(\frac{1}{3}\right)^n \cos(n\omega_0)u(n) \xleftrightarrow{Z} \frac{1-\frac{1}{3}(\cos\omega_0)z^{-1}}{1-\frac{2}{3}(\cos\omega_0)z^{-1}+\frac{1}{9}z^{-2}}$$

ROC $|z| > \frac{1}{3}$.

d)

$$x(n) = \alpha^n u(n) + \alpha^{-n} u(-n) - \delta(n)$$

use the linearity and time-reversal properties

$$X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z} - 1 \quad \frac{1}{2} < |z| < 2$$

$$X(z) = \frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)} \quad \frac{1}{2} < |z| < 2$$

Q5 Find the ROC of the z-transform of each of the following sequences without solving for $X(z)$.

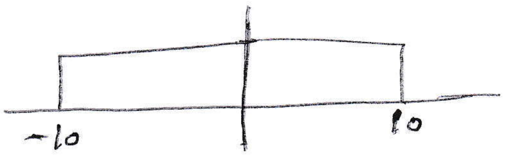
a) $x(n) = \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right] u(n-10)$

b) $x(n) = \begin{cases} 1 & -10 \leq n \leq 10 \\ 0 & \text{otherwise} \end{cases}$

c) $x(n) = 2^n u(-n)$

a) the first sequence is right-sided, the ROC is the exterior of a circle. with a pole at $z = \frac{1}{2}$ coming from the term $\left(\frac{1}{2}\right)^n$, and a pole at $z = \frac{3}{4}$ coming from the term $\left(\frac{3}{4}\right)^n$, it follows that the ROC must be $|z|\frac{3}{4}$

b)




This sequence is finite in length, therefore, the region of convergence is at least $0 < |z| < \infty$. Because $x(n)$ has nonzero values for $n < 0$ and $n > 0$, $z = 0$ and $z = \infty$ are not included within the ROC.

c) Because this sequence is left-sided, the region of convergence is the interior of a circle. with a pole at $z = 2$, it follows that the ROC is $|z| < 2$.

Q6 Find the Z-transform of the sequence $y(n) = \sum_{K=-\infty}^{n} x(K)$

(1) $x(n) = y(n) - y(n-1)$

by shift property of the Z-transform,

$$X(z) = Y(z) - z^{-1}Y(z)$$

$$X(z) = Y(z)\left[1 - z^{-1}\right]$$

$$Y(z) = \frac{1}{1-z^{-1}} X(z)$$

$$\therefore\ y(n) = \sum_{K=-\infty}^{n} x(K) \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}} X(z)$$

(2) $y(n)$ is the convolution of $x(n)$ with a unit step.

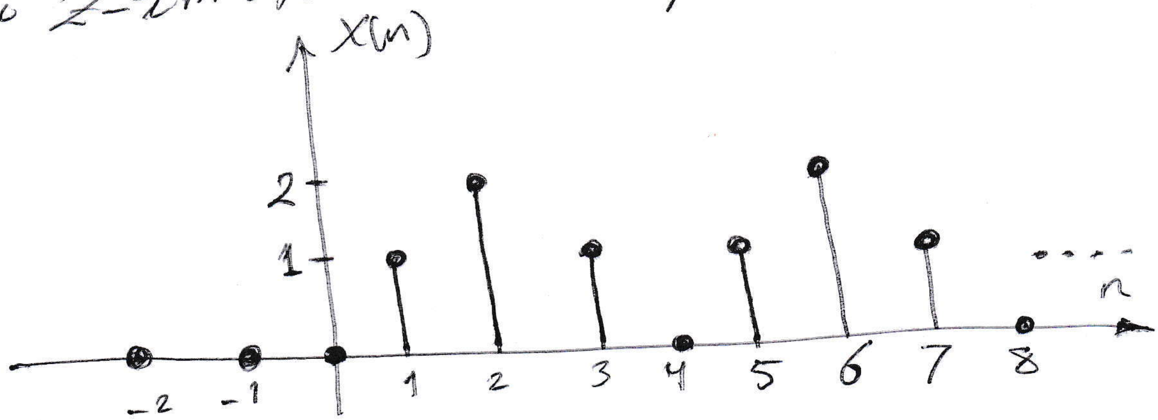
$$y(n) = x(n) * u(n)$$

$$Y(z) = X(z)\, U(z)$$

we have $U(z) = \dfrac{1}{1-z^{-1}}$

$$\therefore\ Y(z) = X(z) \frac{1}{1-z^{-1}}$$

Q7 find the Z-transform of the sequence below.




the period $N = 4$ for $n \geq 0$ and is zero for $n \leq 0$.

$$x(n) = \sum_{k=0}^{\infty} w(n-kN)$$

$$w(n) = \delta(n-1) + 2\delta(n-2) + \delta(n-3)$$

$$W(z) = z^{-1} + 2z^{-2} + z^{-3}$$

$$= z^{-1}\left[1 + 2z^{-1} + z^{-2}\right]$$

$$X(z) = \frac{z^{-1}\left[1 + 2z^{-1} + z^{-2}\right]}{1 - z^{-4}}$$

## Questions about Properties

**Q8** Use the Z-transform to perform the convolution of the following two sequences:-

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & 0 \le n \le 2 \\ 0 & \text{else} \end{cases}$$

$$x(n) = \delta(n) + \delta(n-1) + 4\delta(n-2)$$

$$H(z) = 1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}$$

$$X(z) = 1 + z^{-1} + 4 z^{-2}$$

$$Y(z) = H(z) X(z) = \left(1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}\right)\left(1 + z^{-1} + 4 z^{-2}\right)$$

$$Y(z) = 1 + z^{-1} + 4z^{-2} + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2} + 2 z^{-3} + \frac{1}{4} z^{-2} + \frac{1}{4} z^{-3} + z^{-4}$$

$$Y(z) = 1 + \frac{3}{2} z^{-1} + \frac{19}{4} z^{-2} + \frac{9}{4} z^{-3} + z^{-4}$$

$$\therefore y(n) = \delta(n) + \frac{3}{2}\delta(n-1) + \frac{19}{4}\delta(n-2) + \frac{9}{4}\delta(n-3) + \delta(n-4)$$